TECHNICAL MEMORANDUM

THE NEWTONIAN DIFFUSE METHOD FOR COMPUTING AERODYNAMIC FORCES

BY
W.A. GUSTAFSON

LMSD-5132 28 AUGUST 1958

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ABSTRACT

As part of its continuing support of LMSD projects, the Aerodynamics Department (53-12) has investigated the aerodynamic characteristics of basic shapes in rarified flows. This report contains an analysis of the drag and moment forces on spheres, conical frustums, and cylinders for arbitrary angles of attack. The analysis is based on the assumption of Newtonian flow with diffuse particle reflection. Explicit expressions are derived for quantities of interest.

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INTRODUCTION

At altitudes of about 75 miles and higher, the mean free path between collisions of atmospheric particles is so large that the concepts of continuum fluid mechanics are usually not valid. The computation of aerodynamic forces and moments must then be based upon concepts from the kinetic theory of gases, which provides some understanding of the molecular motion of the gas. The gas particles which make up the atmosphere can be considered in thermal equilibrium in a given region, and the individual particles possess velocities which are described by the Maxwell-Boltzmann velocity distribution function. This function makes it possible to compute the number of particles that strike a surface of unit area per unit of time and the resulting momentum transfer if the nature of the molecular reflection process at the surface is known. It is generally assumed that particles reflect from surfaces either specularly or diffusely, the former meaning that a particle has an equal angle of incidence and angle of reflection, while the latter implies that the reflected particles leave the surface randomly. A particle undergoing specular reflection produces a momentum exchange which is proportional to twice the normal component of velocity, and the computations are relatively simple. For diffuse reflection, it is assumed that an incident particle hits the surface and comes to rest, giving up its total momentum, and then is ejected in a manner that is related to the surface temperature. Diffuse reflection is, of course, a more complicated process than specular reflection and, unfortunately, experimental results pertaining to flight conditions in the upper atmosphere are not available. In reality, both types of reflection may occur, although

The diffuse is probably predominant. Until more data are available, it seems reasonable to assume that the reflection process is completely diffuse. This means that the particles which are ejected from a surface possess a Maxwellian distribution corresponding to the surface temperature.

Free molecular flow consists of superimposing a uniform flow on the thermal motion of the particles and using the superimposed distribution function to calculate the forces arising from the momentum exchange due to collision of particles with the surface of the vehicle. Boundary layers and shock waves do not exist in a clearly distinguishable form, because the mean free path is quite long at high altitude; hence, all intermolecular collisions are neglected.

In the kinetic theory of gases, a quantity called the most probable molecular velocity is defined. This quantity ($C = \sqrt{2RT}$), a sort of average velocity, is proportional to the square root of the temperature, and is related to the speed of sound. The molecular speed ratio is defined as the uniform flow velocity divided by the most probable molecular velocity and is, of course, related to the Mach number. For particular geometries such as the cone, cylinder, and sphere, the drag coefficient as determined from free molecular theory is a function only of the molecular speed ratio, assuming that the vehicle surface has a given temperature. For example, the drag coefficient for a sphere in free molecular flow with diffuse reflection is given in Reference 1* as follows:

$$C_D = \frac{2\ e^{-S^2}}{S\ \sqrt{\pi}} \left(1 + \frac{1}{2\ S^2}\right) + 2\left(1 + \frac{1}{S^2} - \frac{1}{4\ S^4}\right) \ \text{erf}\ S + \frac{2\ \sqrt{\pi}}{3\ S_r} \qquad (1)$$

In Equation (1), S is the molecular speed ratio, and S_r is a molecular speed ratio based on the wall temperature and refers to the reflected particles only. Assuming that only altitudes of over 100 miles are of interest, it is known

* See References

(Reference 2) that the surface temperature of a vehicle is determined almost entirely by radiation phenomena. From the definition of S and S_r, the following is obtained:

$$S_r = \sqrt{\frac{T_i}{T_r}}\, S \tag{2}$$

where T_r is the surface temperature, and T_i is the ambient temperature. Assuming that the surface emissivity is such that $T_r = 550^\circ R$ and noting from the ARDC atmospheric tables that T_i is $1200^\circ R$ or larger above 100 miles, Equation (2), using $T_i = 1200^\circ R$, becomes

$$S_r = 1.47\, S \tag{3}$$

and Equation (1) can be plotted as shown in figure 1 by the upper curve. This curve is a maximum for altitudes above 100 miles, since a higher T_i would cause S_r to increase, thus reducing the contribution of the last term in Equation (1) to the drag coefficient. Taking the limit of Equation (1) as S approaches infinity results in $C_D = 2$. It is apparent that the last term decreases more slowly than the others because it is proportional to S^{-1}, while the others are exponential or of higher exponent. The last term in Equation (1) arises from consideration of the diffuse reflection, and the above simplifications applied to that term are valid only when the surface temperature is primarily controlled by radiation.

Although a free molecular analysis can be applied to any simple shape, sometimes the results cannot be written in terms of elementary functions, and numerical methods must be used. This situation occurs in particular for a cone at an arbitrary angle of attack. It is desirable then to have an approximate method of determining forces and moments on vehicles of various shapes. It is well known that the Newtonian flow concept, assuming that

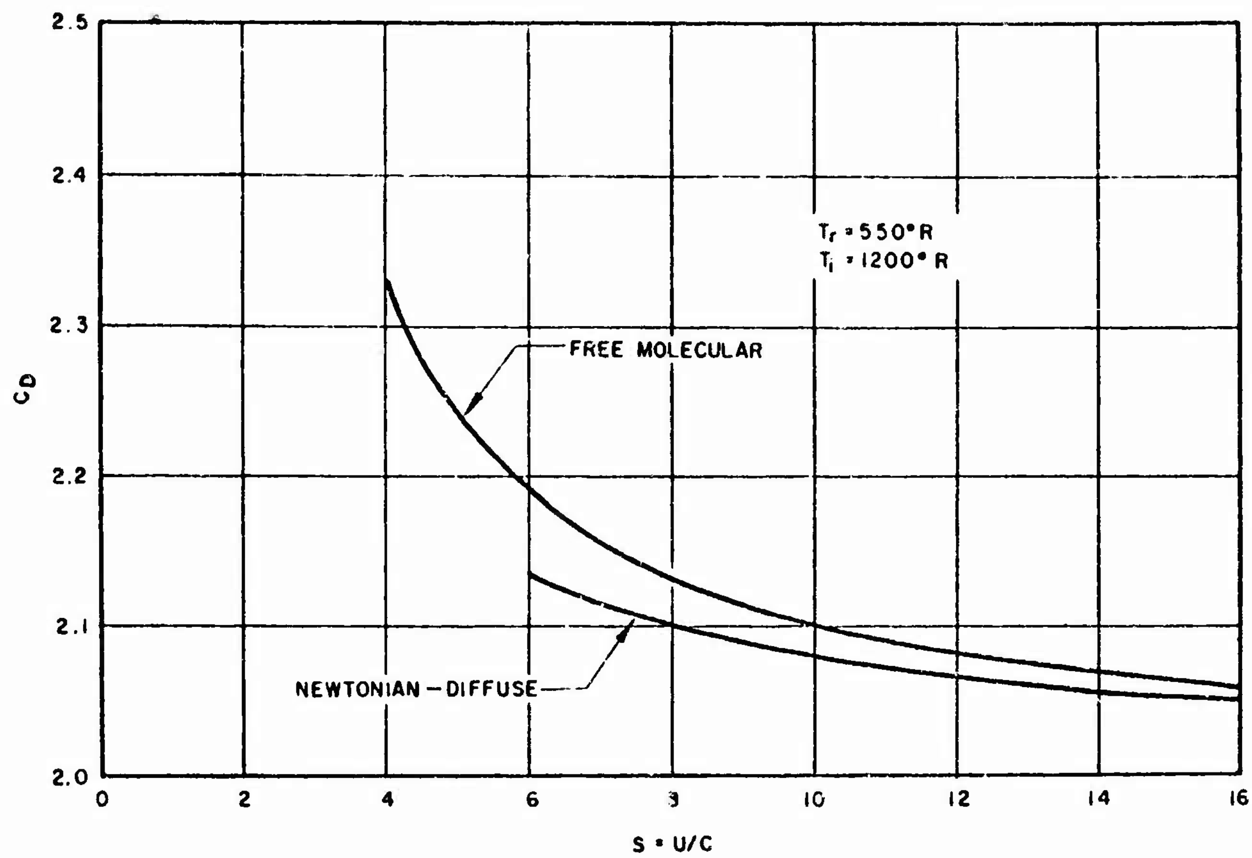

Figure 1 Drag Coefficient vs Molecular Speed Ratio for a Sphere

the particles come to rest on the surface, produces a drag coefficient of 2 for all bodies based on the projected frontal area. This corresponds to the limiting case of Equation (1) as expected, because thermal motion of the particles is neglected, and hence can serve as an approximation as long as S is relatively large. For example, when S = 10, using $C_D = 2$ provides a result which is 4 percent low, as seen from the upper curve of figure 1.

So far nothing has been stated concerning the reflection of particles in Newtonian flow. In the Newtonian hypersonic approximation, it is assumed that the particles give up their normal component of velocity and then slide off the surface tangentially. Newtonian flow with specular reflection imparts twice as muc momentum to the surface as the Newtonian hypersonic approximation. Newtonian flow with completely diffuse reflection is more realistic for the problem under consideration, and the sphere will be analyzed in detail below as an illustration of the method.

The total momentum transferred to an annular element is the drag on that element, and integrating over the spherical surface gives

$$C_D = \frac{D}{\frac{1}{2}\rho U^2 \pi r^2} = \int_0^{\pi/2} 4 \cos\theta \sin\theta \, d\theta = 2 \qquad (4)$$

where ρ is the atmospheric density, U the velocity, and r the radius of the sphere. For diffuse reflection, the kinetic theory of gases provides an expression for the pressure of the ejected particles, assuming that they are in thermal equilibrium with the wall.

$$p_r = \frac{\sqrt{\pi}}{2} C_r m_r \qquad (5)$$

where m_r is the mass leaving a unit surface area per unit of time, assumed equal to the incident mass, and $C_r = \sqrt{2 RT_r}$. Therefore,

$$p_r = \frac{\sqrt{\pi}}{2} \frac{\rho U^2}{S_r} \cos\theta$$

and taking the component of this term in the drag direction and integrating over the surface gives

$$D_r = \frac{\sqrt{\pi}}{2} \; \frac{\rho U^2}{S_r} \int_0^{\pi/2} 2\,\pi r^2 \cos^2 \theta \sin\theta \, d\theta \tag{6}$$

or

$$C_D = \frac{2\sqrt{\pi}}{3\,S_r}$$

The total drag coefficient is then the sum of Equations (4) and (6)

$$C_D = 2 + \frac{2\sqrt{\pi}}{3\,S_r} \tag{7}$$

This equation is plotted in figure 1, and the curve is labeled Newtonian-diffuse. The curve is a fairly good approximation of the free molecular result, and even at S = 6 there is only a 2.8-percent discrepancy. This is because the reflection term does not decrease rapidly as S becomes large. The utility of the Newtonian-diffuse analysis can be appreciated by studying figure 2, which is a plot of altitude against the molecular speed ratio corresponding to the velocity of a satellite in circular orbit at each altitude. As can be seen, the Newtonian-diffuse approximation provides a result for the drag coefficient of a sphere which is at most about 3 percent in error throughout the altitude range in which aerodynamic forces on satellites are of significance. At the lower altitudes, this error becomes somewhat less, and the Newtonian-diffuse theory should be adequate for most purposes even when the aerodynamic moment is a critical factor. The error associated with the Newtonian-Diffuse theory is different for other bodies and is discussed in the Appendix.

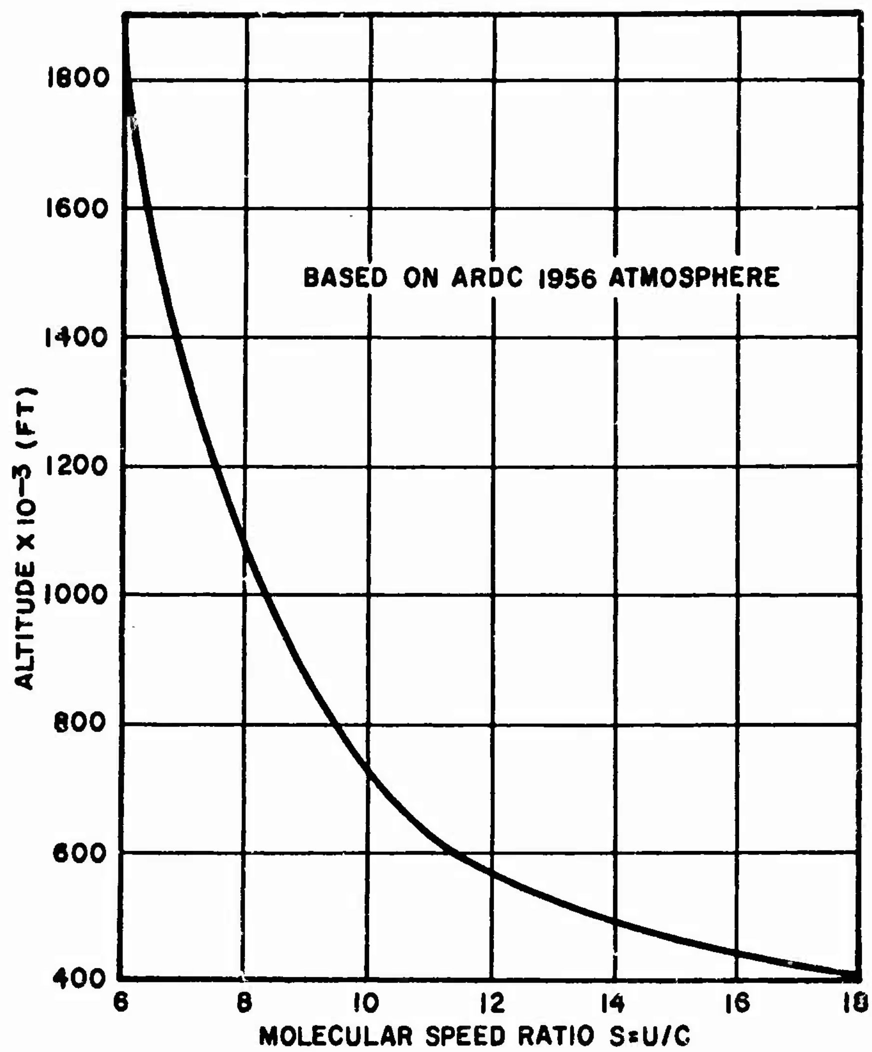

Figure 2 Molecular Speed Ratio vs Altitude

The following sections provide formulas for computing the aerodynamic moment and the drag forces on shapes which usually are of interest for orbiting vehicles.

SECTION II
ANALYSES FOR CONE SHAPE

The analysis of drag and moment on a cone is divided into two parts, one for the angle of attack greater than the semi-vertex angle, and one for the angle of attack less than the semi-vertex angle. The moments due to the incidence and reflection of particles from the surface are also considered separately, and the results given below.

2.1 INCIDENT MOMENT $\alpha < \delta$

Figure 3 illustrates the geometry of the problem and the notation used in the following equations. The force on a unit of surface area is in the free stream direction and is given by

$$\rho U^2 \ell_x \tag{8}$$

where ℓ_x is the direction cosine between the free stream and the inward normal at any point on the cone:

$$\ell_x = \sin \delta \cos \alpha + \sin \alpha \cos \delta \cos \phi \tag{9}$$

The moment arm from this force to an arbitrary vehicle center of gravity located on the cone axis a distance L from the vertex is

$$(L - y) \sin \alpha - y \tan \delta \cos \alpha \cos \phi \tag{10}$$

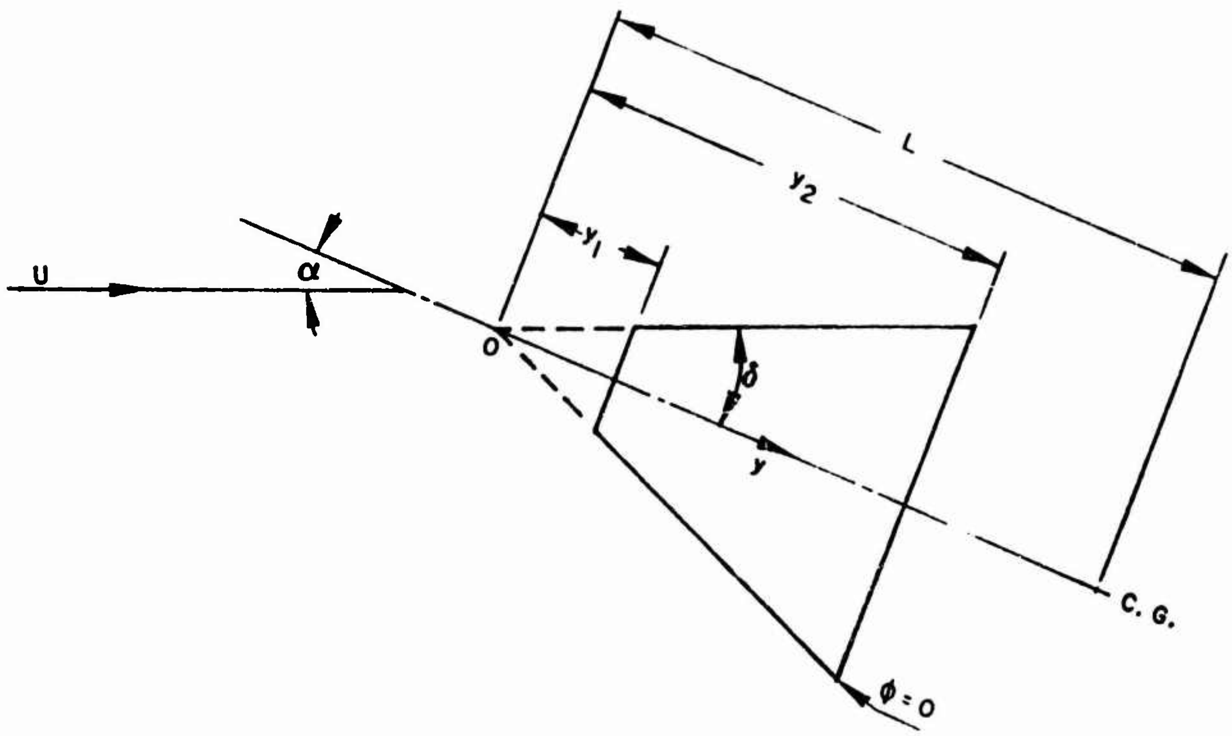

Figure 3 Geometry and Notation for Cone

and an element of surface area is

$$dA = \frac{\tan \delta}{\cos \delta} \, y \, dy \, d\phi \tag{11}$$

Combining the above four quantities gives

$$M_{CG} = 2\rho U^2 \int_0^{\pi}\int_{y_1}^{y_2} \ell_x \left[(L - y) \sin \alpha - y \tan \delta \cos \alpha \cos \phi \right] \frac{\tan \delta}{\cos \delta} \, y \, dy \, d\phi \tag{12}$$

and carrying out the indicated integration provides the following result:

$$M_{CG} = 2\rho U^2 \pi \tan^2 \delta \sin \alpha \cos \alpha \left[L\left(\frac{y_2^2 - y_1^2}{2}\right) - \frac{3}{2}\left(\frac{y_2^3 - y_1^3}{3}\right)\right] \qquad (13)$$

where $y_1 \neq 0$ for a cone frustum.

2.2 DIFFUSE REFLECTION MOMENT $\alpha < \delta$

The reflection process exerts a pressure on the surface which is given by

$$p_r = \frac{\sqrt{\pi}}{2}\sqrt{\frac{T_r}{T_1}}\, \ell_x \frac{\rho U^2}{S} \qquad (14)$$

where the incident mass is equal to

$$m_1 = \rho U \ell_x \qquad (15)$$

The pressure is divided into components parallel and normal to the cone axis, giving a combined moment arm equal to

$$L \cos \delta \cos \phi - \frac{y \cos \phi}{\cos \delta} \qquad (16)$$

and the moment is expressed as follows:

$$M_{CG} = \sqrt{\pi}\sqrt{\frac{T_r}{T_1}}\frac{\rho U^2}{S}\int_0^{\pi}\int_{y_1}^{y_2} \ell_x \left(L \cos \delta - \frac{y}{\cos \delta}\right) \cos \phi \frac{\tan \delta}{\cos \delta}\, y\, dy\, d\phi \qquad (17)$$

The result of the integration is

$$M_{CG} = \sqrt{\pi}\ \sqrt{\frac{T_r}{T_1}}\ \frac{\rho U^2}{S}\ \frac{\pi}{2}\frac{\tan\delta}{\cos\delta}\ \sin\alpha\left[L\cos^2\delta\left(\frac{y_2^2 - y_1^2}{2}\right) - \left(\frac{y_2^3 - y_1^3}{3}\right)\right] \tag{18}$$

2.3 INCIDENT MOMENT $\alpha > \delta$

The results can be obtained for this case in a manner similar to that for the case $\alpha < \delta$ except that now part of the cone is shielded from the free stream. This area is defined by the locus of points at which the free stream becomes tangent to the cone and is found by setting Equation (9) equal to zero, solving for ϕ, and using this as the limit of the ϕ integration. The final result for this case is

$$\begin{aligned} M_{CG} &= 2\rho U^2\left(\frac{y_2^2 - y_1^2}{2}\right) L \sin\alpha\ \frac{\tan\delta}{\cos\delta}\ A \\ &\quad - 2\rho U^2\left(\frac{y_2^3 - y_1^3}{3}\right)\frac{\tan\delta}{\cos\delta}\left[A\sin\alpha + B\cos\alpha\tan\delta\right] \end{aligned} \tag{19}$$

where $A = \cos\alpha\sin\delta\left[\pi - \cos^{-1}(\cot\alpha\tan\delta)\right] + \sin\alpha\cos\delta\sin\left[\cos^{-1}(\cot\alpha\tan\delta)\right]$

and $B = \cos\alpha\sin\delta\sin\left[\cos^{-1}(\cot\alpha\tan\delta)\right]$

$$+ \sin\alpha\cos\delta\left\{\frac{\pi}{2} - \frac{1}{2}\cos^{-1}(\cot\alpha\tan\delta) - \frac{1}{4}\sin\left[2\cos^{-1}(\cot\alpha\tan\delta)\right]\right\}$$

2.4 DIFFUSE REFLECTION MOMENT $\alpha > \delta$

The moment about an arbitrary point located on the line of the cone axis is

$$M_{CG} = \sqrt{\pi} \sqrt{\frac{T_r}{T_i}} \frac{\rho U^2}{S} \frac{\tan \delta}{\cos \delta} B \left[\left(\frac{y_2^2 - y_1^2}{2} \right) L \cos \delta - \left(\frac{y_2^3 - y_1^3}{3} \right) \frac{1}{\cos \delta} \right] \tag{20}$$

where B is defined above.

2.5 DRAG FORCE $\alpha < \delta$

The drag force is made up of contributions from the incident and reflected particles. The result below includes both contributions.

$$D = 2\rho U^2 \left(\frac{y_2^2 - y_1^2}{2} \right) \pi \tan^2 \delta \cos \alpha$$

$$+ \sqrt{\pi} \sqrt{\frac{T_r}{T_i}} \frac{\rho U^2}{S} \left(\frac{y_2^2 - y_1^2}{2} \right) \frac{\tan \delta}{\cos \delta} \left[\sin^2 \delta \cos^2 \alpha + \frac{1}{2} \sin^2 \alpha \cos^2 \delta \right] \tag{21}$$

If the drag coefficient is desired, Equation (21) must be divided by $\frac{1}{2} \rho U^2 A$, where A is some particular reference area. Usually it is most convenient to use the base area for conical shapes, and hence for this case

$$A = \pi \left(y_2^2 - y_1^2 \right) \tan^2 \delta$$

2.6 DRAG FORCE $\alpha > \delta$

The drag force for conditions when $\alpha > \delta$, including the incident and diffuse contributions, is given below.

$$D = 2\rho U^2 \left(\frac{y_2^2 - y_1^2}{2}\right) \frac{\tan\delta}{\cos\delta} \Bigg\{ \cos\alpha \sin\delta \left[\pi - \cos^{-1}(\cot\alpha \tan\delta)\right] \left[1 + \frac{\sqrt{\pi}}{2S}\sqrt{\frac{T_r}{T_i}} \cos\alpha \sin\delta\right]$$

$$+ \sin\alpha \cos\delta \sin\left[\cos^{-1}(\cot\alpha\tan\delta)\right]\left(1 + 2\cos\alpha\sin\delta \frac{\sqrt{\pi}}{2S}\sqrt{\frac{T_r}{T_i}}\right) \qquad (22)$$

$$+ \frac{\sqrt{\pi}}{2S}\sqrt{\frac{T_r}{T_i}} \sin^2\alpha \cos^2\delta \left[\frac{\pi}{2} - \frac{1}{2}\cos^{-1}(\cot\alpha\tan\delta) - \frac{1}{4}\sin\left|2\cos^{-1}(\cot\alpha\tan\delta)\right.\right]\Bigg\}$$

SECTION III
ANALYSES FOR CYLINDRICAL SHAPE

The geometry and notation for the cylinder are shown in figure 4, and the essential steps in the derivation of the equations are given below.

3.1 INCIDENT MOMENT

The direction cosine between the free stream and an inward normal of the surface is

$$\ell_x = \sin \alpha \cos \Phi \tag{23}$$

The moment arm from an arbitrary point on the cylinder axis to an element of the surface is

$$(L + y) \sin \alpha - r \cos \alpha \cos \Phi \tag{24}$$

and hence the moment can be written as

$$M_{CG} = 2\rho U^2 \int_0^{y_2} \int_0^{\pi/2} \left[(L + y) \sin \alpha - r \cos \alpha \cos \Phi \right] \sin \alpha \cos \Phi \; r \, d\Phi \, dy \tag{25}$$

The result of the integration provides the following result:

$$M_{CG} = 2\rho U^2 r y_2 \sin \alpha \left[\left(L + \frac{y_2}{2} \right) \sin \alpha - \frac{r\pi}{4} \cos \alpha \right] \tag{26}$$

If it is desired to compute the moment about some point located inside the cylinder a distance L from the origin of the y coordinate, then a negative sign is inserted in front of L in Equation (26). The ends of the cylinder are not included in the analysis.

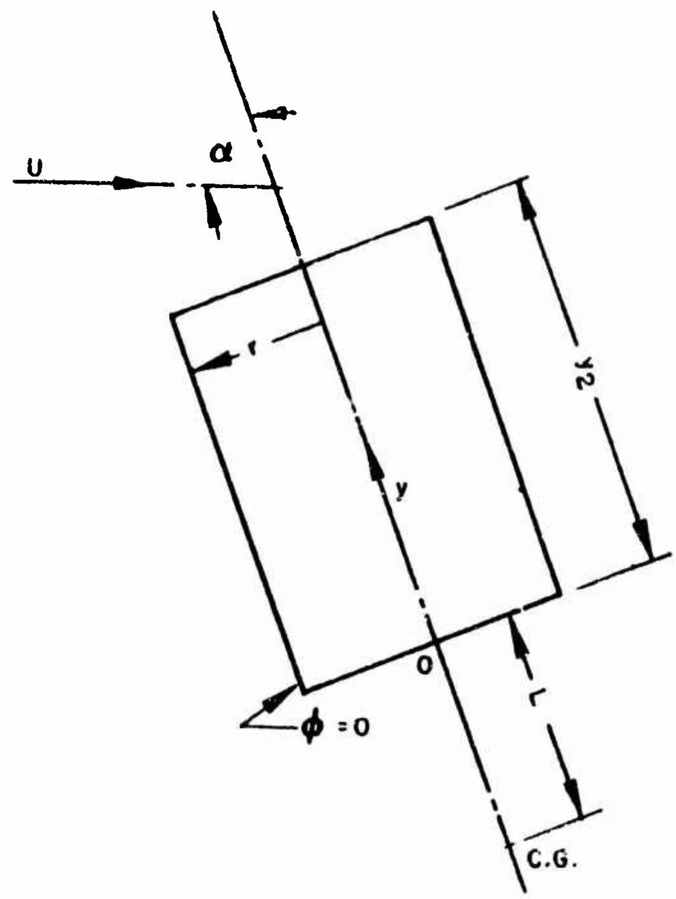

Figure 4 Geometry and Notation for Cylinder

3.2 DIFFUSE REFLECTION MOMENT

Using the definition of the pressure due to reflection as given in Equation (14), the following equation results for the moment about an arbitrary point on the cylinder axis:

$$M_{CG} = \sqrt{\pi} \sqrt{\frac{T_r}{T_i}} \frac{\rho U^2}{S} \int_0^{y_2} \int_0^{\pi/2} (L + y) \sin \alpha \cos^2 \phi \; r \, d\phi dy \qquad (27)$$

or

$$M_{CG} = \frac{\pi}{4}^{3/2} \sqrt{\frac{T_r}{T_i}} \frac{\rho U^2}{S} r y_2 \sin \alpha \left(L + \frac{y_2}{2} \right) \qquad (28)$$

As before, the moment about a point inside the cylinder is obtained by putting a negative sign in front of L. No contribution from the ends of the cylinder is included in the above computations.

3.3 DRAG FORCE

The drag force equation given below consists of contributions from the incident and reflected particles.

$$D = 2\rho U^2 \, r y_2 \sin\alpha \left[1 + \frac{\pi^{3/2}}{8\,S} \sqrt{\frac{T_r}{T_1}} \; \sin\alpha \right] \qquad (29)$$

The ends of the cylinder are neglected in Equation (29), as can be observed since the drag is zero at zero angle of attack.

SECTION IV
ANALYSES FOR SPHERICAL SHAPE

Since the net momentum exchange on a sphere produces only a drag force, those computations in which an entire sphere is involved are relatively simple. The drag coefficient is given by Equation (7), based on the projected frontal area, and hence the moment of the sphere about any other point amounts to determining the moment arm and multiplying by the drag.

In a number of cases where a vehicle is composed of a combination of cones, cylinders, or spheres, a cone or a cylinder may be capped by a part of a sphere or a hemisphere. In the paragraphs below, the method of computation for such cases will be presented.

4.1 INCIDENT MOMENT $\alpha < \pi/2 - \theta_1$

Figure 5 illustrates the notation which is used. When a frustum of a cone is capped by a portion of a sphere, the semi-vertex angle of the cone defines the angle θ_1, which determines the required part of the spherical cap needed to cover the cone. When the angle of attack is less than the angle $\pi/2 - \theta_1$, then the free stream strikes the entire surface of the spherical cap, and the following analysis is applicable. The direction cosine between the free stream and an inner normal is

$$\ell_x = \cos\theta \cos\alpha + \sin\theta \sin\alpha \cos\phi \tag{30}$$

and the moment arm is

$$\bar{R} = \ell \sin\alpha + (r \cos\theta \tan\alpha - r \sin\theta \cos\phi) \cos\alpha \tag{31}$$

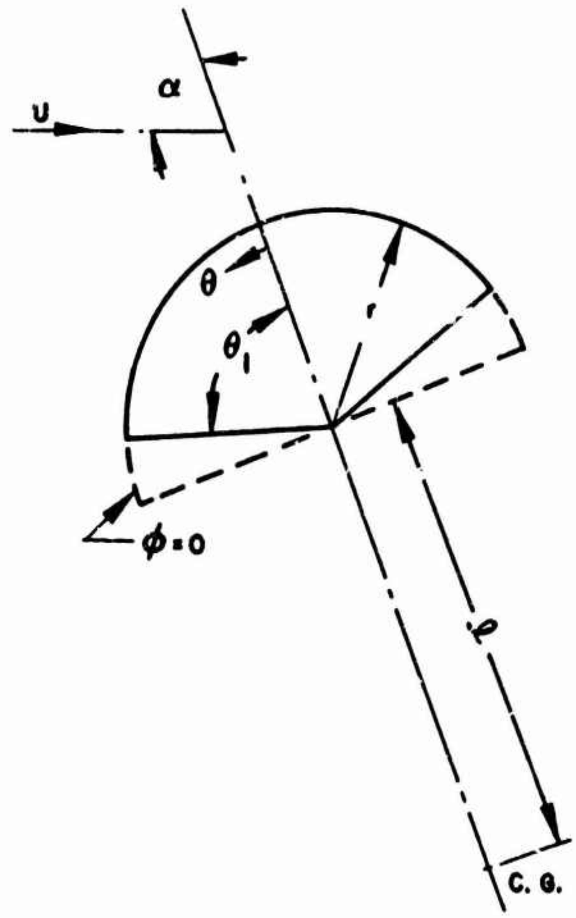

Figure 5 Geometry and Notation for Sphere

An element of area on the spherical surface can be written as

$$dA = r^2 \sin\theta \, d\theta d\phi \tag{32}$$

and hence the moment about an arbitrary point on the polar axis is

$$M_{CG} = 2\rho U^2 \int_0^{\theta_1}\int_0^{\pi} \ell_x \, \overline{R} \, dA \tag{33}$$

Carrying out the indicated integration provides the following result:

$$M_{CG} = 2\rho U^2 \pi r^2 \sin\alpha \cos\alpha \sin^2\theta_1 \left(\frac{\ell}{2} + \frac{r}{2}\cos\theta_1\right) \tag{34}$$

4.2 DIFFUSE REFLECTION MOMENT $\alpha < \pi/2 - \theta_1$

Using the definition of the pressure, Equation (14), and dividing the pressure into components perpendicular and parallel to the polar axis yields the following result:

$$M_{CG} = \frac{\sqrt{\pi}}{S}\sqrt{\frac{T_r}{T_i}}\,\rho U^2 \ell \int_0^{\theta_1}\int_0^{\pi} \ell_x \sin\theta \cos\phi \, r^2 \sin\theta \, d\theta \, d\phi \qquad (35)$$

Carrying out the integration gives

$$M_{CG} = \frac{\sqrt{\pi}}{S}\sqrt{\frac{T_r}{T_i}}\,\frac{\rho U^2}{2}\,\ell \pi r^2 \sin\alpha \left[\frac{2}{3} - \frac{\cos\theta_1}{3}\left(\sin^2\theta_1 + 2\right)\right] \qquad (36)$$

4.3 INCIDENT MOMENT $\alpha > \pi/2 - \theta_1$

For this case, the integration must be done in two parts, because over part of the body the flow hits elements of area for all values of ϕ, while on the rest of the body the shielding effect does not allow impingement of the free stream. The moment is thus given by

$$M_{CG} = 2\rho U^2 \left\{ \int_{\theta=0}^{\pi/2-\alpha}\int_{\phi=0}^{\pi} \ell_x \bar{R}\, dA + \int_{\pi/2-\alpha}^{\theta_1}\int_0^{\pi-\cos^{-1}(\cot\alpha\,\cot\theta)} \ell_x \bar{R}\, dA \right\} \qquad (37)$$

The integration is very lengthy, although it can be done, with the result below.

$$
\begin{aligned}
M_{CG} = 2\rho U^2 r^2 \Bigg\{ & \frac{\pi r}{4} \sin\alpha \cos\alpha \left[-\frac{1}{3} + 2\sin\alpha \cos^2\alpha \right] \\
& + \frac{\ell\pi}{4} \sin\alpha \cos\alpha \left[\frac{\cos\alpha}{2} + 2\sin^2\theta_1 \right] + \frac{\ell\pi}{4} \sin^3\alpha \\
& + \frac{\sin\alpha}{6} \cos\alpha \cos^{-1}(\cot\alpha \cot\theta_1) \Big[-3\ell \sin^2\theta_1 + 2r\cos^3\theta_1 \\
& \qquad - 3r\cos\theta_1 \left(1 - \frac{1}{3}\cos^2\theta_1 \right) \Big] \\
& - \frac{\ell}{4} \sin\alpha \cos^2\alpha \sin^{-1}\left[1 + 2\cot^2\alpha - 2\csc^2\alpha \sin^2\theta_1 \right] \\
& + \frac{r}{6} \sin^2\alpha \left(2\sin^2\alpha - \cos^2\alpha \right) \sqrt{(\sin^2\theta_1 \csc^2\alpha - \cot^2\alpha)^3} \\
& + \frac{\ell}{2} \sin^3\alpha \tan^{-1}\sqrt{\frac{\cos^2\theta_1}{\sin^2\theta_1 - \cos^2\alpha}} - \frac{r}{6} \sin\alpha \cos\alpha \sin^{-1}\left[-2\cos^2\alpha \csc^2\theta_1 + 1 \right] \\
& + \frac{r}{2} \sin\alpha \cos^3\alpha \sqrt{\frac{1 - \cot^2\alpha \cot^2\theta_1}{\cot^2\alpha \csc^2\theta_1}} + \frac{r\pi}{2} \sin\alpha \cos\alpha \Big[\sin^2\theta_1 \cos\theta_1 \\
& \qquad - \sin\alpha \cos^2\alpha \Big] \\
& - \frac{r}{3} \sin\alpha \cos\alpha \tan^{-1}\sqrt{\frac{1 - \cot^2\alpha \cot^2\theta_1}{\cot^2\alpha \csc^2\theta_1}} \\
& - \frac{\ell}{2} \sin^2\alpha \cos\theta_1 \sqrt{\sin^2\theta_1 \csc^2\alpha - \cot^2\alpha}
\end{aligned}
\qquad (38)
$$

This equation can be checked at two points. When $\alpha = \pi/2 - \theta_1$, this equation must reduce to Equation (34) with $\alpha = \pi/2 - \theta_1$ in that equation also. When $\alpha = \theta_1 = \pi/2$, Equation (38) must reduce to the moment of a hemisphere about a point a distance ℓ from the center of the sphere which is given below:

$$M_{CG} = 2\rho U^2 \left(\frac{\ell\pi}{4} + \frac{r}{3}\right) r^2 \qquad (39)$$

These conditions are satisfied by Equation (38) if the terms involving the arc tangent are handled properly. In each of those terms, the negative root of the radical is taken, and the angle determined is a negative angle. For example, if the quantity inside the radical is one, then the negative root is minus one, and measuring the angle in the negative direction gives $\tan^{-1}(-1) = -\pi/4$. Also principal values of the arc sine are to be taken.

4.4 DIFFUSE REFLECTION MOMENT $\alpha > \pi/2 - \theta_1$

The result for this case is

$$\begin{aligned} M_{CG} = \frac{\sqrt{\pi}}{S} \sqrt{\frac{T_r}{T_1}}\, \rho U^2 \ell\, r^2 \Bigg\{ & \frac{\pi}{3} \sin\alpha - \frac{\pi}{6} \sin\alpha \cos\theta_1 \left(\sin^2\theta_1 + 2\right) \\ & + \frac{1}{6} \sin^2\alpha \cos\alpha \sqrt{\left(\sin^2\theta_1 \csc^2\alpha - \cot^2\alpha\right)^3} \qquad (40) \\ & + \frac{1}{3} \sin\alpha \tan^{-1} \sqrt{\frac{1-\cot^2\alpha \cot^2\theta_1}{\cot^2\alpha \csc^2\theta_1}} \\ + \frac{1}{2} \sin\alpha \cos\theta_1 \left(1 - \frac{1}{3}\cos\theta_1\right) & \cos^{-1}(\cot\alpha \cot\theta_1) - \frac{1}{6} \sin\alpha \cos^2\alpha \sqrt{\frac{1-\cot^2\alpha \cot^2\theta_1}{\cot^2\alpha \csc^2\theta_1}} \Bigg\} \end{aligned}$$

Here again, in the term involving arc tangent, the negative root is taken, and the result is taken as a negative angle.

4.5 DRAG FORCE $\alpha < \pi/2 - \theta_1$

The drag force, including both the incidence and diffuse reflection of particles from the surface, is

$$D = \rho U^2 \pi r^2 \left\{ \cos\alpha \sin^2\theta_1 + \frac{\sqrt{\pi}}{S}\sqrt{\frac{T_r}{T_i}} \left[\frac{1}{3}\left(\sin^2\alpha + \cos^2\alpha\right)\left(1 - \cos\theta_1\right) - \frac{1}{6}\sin^2\theta_1 \cos\theta_1 \left(\sin^2\alpha - 2\cos^2\alpha\right) \right] \right\} \quad (41)$$

4.6 DRAG FORCE $\alpha > \pi/2 - \theta_1$

The drag force for this case is rather lengthy, and both the incident and diffuse contributions are included.

$$D = 2\rho U^2 r^2 \left\{ \frac{\pi}{2} \cos\alpha \sin^2\theta_1 - \frac{\cos\alpha}{2} \sin^2\theta_1 \cos^{-1}\left(\cot\alpha \cot\theta_1\right) \right.$$

$$+ \frac{\pi}{8}\left(\cos^2\alpha + 2\sin^2\alpha\right) - \frac{\cos^2\alpha}{4} \sin^{-1}\left[1 + 2\cot^2\alpha - 2\csc^2\alpha \sin^2\theta_1\right]$$

$$- \frac{\sin\alpha}{2} \cos\theta_1 \sqrt{\sin^2\theta_1 \csc^2\alpha - \cot^2\alpha}$$

$$\left. + \frac{\sin^2\alpha}{2} \tan^{-1} \sqrt{\frac{\cos^2\theta_1}{\sin^2\theta_1 - \cos^2\alpha}} \right\}$$

$$+ \frac{\sqrt{\pi}}{S} \sqrt{\frac{T_r}{T_i}}\, \rho U^2 r^2 \left\{ -\frac{\pi}{4}\cos^2\alpha + \frac{\pi}{6}\sin^2\alpha \left(2 + 3\cos^3\theta_1\right) \right.$$

$$- \frac{\pi}{3}\cos^3\theta_1 + \frac{\cos^2\alpha}{3} \cos^3\theta_1 \cos^{-1}\left(\cot\alpha \cot\theta_1\right)$$

$$+ \frac{\sin^2\alpha}{2} \cos\theta_1 \left(1 - \frac{1}{3}\cos\theta_1\right) \cos^{-1}\left(\cot\alpha \cot\theta_1\right)$$

$$- \frac{\cos^2\alpha}{6} \sin^{-1}\left[-2\cos^2\alpha \csc^2\theta_1 + 1\right]$$

$$+ \frac{\cos^2\alpha}{6}\left(2\cos^2\alpha - \sin^2\alpha\right) \sqrt{\frac{1 - \cot^2\alpha \cot^2\theta}{\cot^2\alpha \csc^2\theta_1}}$$

$$+ \frac{1}{2}\sin^3\alpha \cos\alpha \sqrt{\left(\sin^2\theta_1 \csc^2\alpha - \cot^2\alpha\right)^3} - \frac{\pi}{2}\sin^2\alpha \cos\theta_1$$

$$\left. + \frac{1}{3}\sin^2\alpha \tan^{-1} \sqrt{\frac{1 - \cot^2\alpha \cot^2\theta_1}{\cot^2\alpha \csc^2\theta_1}} \right\}$$

SECTION V
CONCLUSIONS

In some of the results for the drag force, the contribution of the reflected particles results in cumbersome equations which are difficult to evaluate. Since the drag is mainly of interest in determining orbit lifetimes, it may be entirely satisfactory to use the value of 2 for the drag coefficient based on the projected frontal area. However, the effect of diffuse reflection on aerodynamic moments is not always small in relation to the total moment, and it should not be neglected if attitude control of the vehicle is a critical design factor.

All of the equations above are valid for any angle of attack. When they are applied to a particular composite body, however, the orientation of the coordinate axes must conform to the system used in the derivation of the equations.

SECTION VI
REFERENCES

The following articles are referenced in the body of this report:

1. J.R. Stalder and V.J. Zurick, "Theoretical Aerodynamic Characteristics of Bodies in a Free-Molecular-Flow Field," NACA TN 2423, July 1951.

2. J.R. Stalder and D. Jukoff, "Heat Transfer to Bodies Traveling at High Speed in the Upper Atmosphere," <u>Jour. of Aeronautical Sciences,</u> vol. 15, July 1948.

APPENDIX

The equations presented in this report based on the Newtonian-diffuse theory are of course approximate in that the thermal motion of the molecules in the free stream is neglected. This approximation introduces an error into the results which depends on the body shape and varies with the angle of attack for a particular shape. An illustration of this can be seen by inspecting the free molecular equations for lift and drag on a flat plate and observing that the important parameter is the product of the molecular speed ratio and the sine of the angle of attack. The Newtonian-diffuse equations for lift and drag on a flat plate can be easily obtained, and a comparison with the free molecular equations indicates that the error increases with decreasing angle of attack for a given value of S. Hence, the resulting equations presented in this report for the cylinder can be expected to show a similar tendency. Accurate results for a cylinder can be obtained by considering free molecular theory which accounts for the free stream thermal motion of atmospheric particles. The analysis of the aerodynamic moment is presented in the following paragraphs, and the result can be expressed in terms of Bessel functions which are easily evaluated. Applying a similar analysis to the cone or spherical cap results in expressions requiring numerical integration. Therefore, the Newtonian-diffuse equations for those two bodies represent the only analytical result available for the flow regime under consideration.

The notation and coordinate system used below corresponds to that used by Talbot on page 458 in the June 1957 issue of the Journal of the Aeronautical Sciences. Talbot's y axis is used for a local coordinate system and does not appear in the results; thus, no confusion will result by using the y

axis as the cylinder axis, as illustrated in Figure 4 of this report. Talbot defines the N and T directions to be normal and along the cylinder axis respectively and lying in the plane of the pitching moment, which is the quantity to be determined. Taking the T direction to be opposite from that of Talbot, the components of the pressure and shear forces in the N and T directions can be written in terms of the angle of attack β and the polar angle ϕ as follows:

$$\frac{p_N}{\frac{1}{2}\rho U^2} = \left(\frac{\sin\beta\sin\phi}{S\sqrt{\pi}} + \frac{1}{2S^2}\sqrt{\frac{T_r}{T_1}}\right)\sin\phi\, e^{-\xi^2\sin^2\phi}$$

$$+ \sin\phi\left(\frac{1}{2S^2} + \sin^2\beta\sin^2\phi + \frac{\sqrt{\pi}}{2S}\sin\beta\sin\phi\sqrt{\frac{T_r}{T_1}}\right)\left[1 + \mathrm{erf}\,(\xi\sin\phi)\right] \tag{1A}$$

$$\frac{\tau_N}{\frac{1}{2}\rho U^2} = -\frac{\sin\beta\cos^2\phi}{S\sqrt{\pi}}\left\{e^{-\xi^2\sin^2\phi} + \sqrt{\pi}\,\xi\sin\phi\left[1 + \mathrm{erf}\,(\xi\sin\phi)\right]\right\} \tag{2A}$$

$$\frac{\tau_T}{\frac{1}{2}\rho U^2} = \frac{\cos\beta}{S\sqrt{\pi}}\left\{e^{-\xi^2\sin^2\phi} + \sqrt{\pi}\,\xi\sin\phi\left[1 + \mathrm{erf}\,(\xi\sin\phi)\right]\right\} \tag{3A}$$

where $\xi = S\sin\beta$ and $p_T = 0$. Hence, the pitching moment about a point on the cylinder axis a distance L from the origin of the y coordinate is given by the following expression:

$$M_{CG} = 2\int_{-\pi/2}^{\pi/2}\int_{0}^{y_2}\left[(p_N - \tau_N)(y + L) - \tau_T\, r\sin\phi\right] r\, dy\, d\phi \tag{4A}$$

The integrals in (4A) can be easily evaluated giving the final result:

$$\frac{M_{CG}}{\frac{1}{2}\rho U^2} = 2ry_2 \sqrt{\pi}\left(\frac{y_2}{2} + L\right)\left\{\frac{\pi}{4S} \sin\beta \sqrt{\frac{T_r}{T_1}}\right.$$

$$\left. + 3 \sin^2\beta\, e^{-\frac{\xi^2}{2}} \left[\left(\frac{1}{2\xi} + \frac{\xi}{3}\right) I_o\left(\frac{\xi^2}{2}\right) + \left(\frac{\xi}{3} + \frac{1}{6\xi}\right) I_1\left(\frac{\xi^2}{2}\right)\right]\right\} - \pi r^2 y_2 \sin\beta \cos\beta \tag{5A}$$

where $I_o\left(\frac{\xi^2}{2}\right)$ and $I_1\left(\frac{\xi^2}{2}\right)$ are modified Bessel functions of the first kind and zero and first order respectively. When ξ becomes very large or when terms of the order $1/\xi^2$ can be neglected, then (5A) reduces to the Newtonian-diffuse result given by the sum of equations (26) and (28). A comparison of these two results shows that by accounting for the free stream thermal motion, the first term of (26) goes into the more complicated second term of (5A), which is a function of S sin β. Therefore, it is evident that the Newtonian-diffuse result will be in error to an increasing extent as the angle of attack decreases at a particular molecular speed ratio.